J. Fluid Mech. (1977), *vol.* 79, *part* 3, *pp.* 499–524
Printed in Great Britain

The wave system attached to a slender body in a supersonic relaxing gas stream. Basic results: the cone

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(Received 11 June 1976)

The results of the linear theory for the flow of a supersonic relaxing gas past a slender body of revolution are analysed in regions where its predictions of wavelet position begin to break down. In this way new variable systems can be found which make it possible to discuss the correct nonlinear wave behaviour far from the body. The situation depends upon three especially important parameters, namely the thickness ratio ϵ of the body, the ratio δ of relaxing-mode energy to thermal energy and the ratio λ of a relaxation length to a typical body length. After establishing general results from the linear theory, the conical body is treated in some detail. This makes it possible to demote λ as an important parameter, although its restoration does prove useful at one point in the analysis, and results are derived for shock-wave behaviour when ord $1 \geqslant \delta >$ ord ϵ^4, $\delta =$ ord ϵ^4 and $\delta <$ ord ϵ^4. In the first range of δ fully dispersed waves are essential, although they are fully established only at great distances from the cone; in the second range of δ partly dispersed waves seem to be the most likely to appear, and in the third range relaxation effects are second-order modifications of a basically frozen-flow field. Practical situations may well fall into the first of these categories.

1. Introduction

The propagation of disturbances through a gas can be significantly affected by the presence within the gas of a mode of energy storage with a finite, non-zero, relaxation time. One-dimensional unsteady and two-dimensional steady configurations have been studied in the past (a reasonably up-to-date account can be found in the book by Clarke & McChesney 1976), but although some early work exists on the linear theory of supersonic axially symmetric flow, there has been no attempt to date to analyse the nonlinear far field of such a flow. An interesting study of the shock wave near the nose of a body of revolution has been carried out by Chou & Chu (1971) but their method, based on the use of characteristic parameters, cannot be used to examine the truly distant flow field. The probable relevance of relaxation effects to the sonic boom (Hodgson & Johannesen 1971) is certainly one reason for re-opening the question of wave

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propagation through a relaxing gas, and the axisymmetric configuration is basic in this case just as it is in the now highly developed analyses of wave propagation from real aircraft through non-relaxing, but otherwise real, atmospheres.

A feature of the vibrational relaxation of atmospheric oxygen and nitrogen is the small proportion of the total thermal energy that is contributed by the relaxing mode (Hodgson & Johannesen 1971). However, in an axisymmetric configuration the geometric attenuation of the disturbance field means that the energy of the perturbed gas motion is also exceedingly small far from the body and one must therefore not dismiss relaxation effects as simply 'higher-order corrections'; in particular they prove to be of first importance in their effect on the very weak shock waves that exist in supersonic flow past a slender body. The small relaxing-mode energy means that there is especial interest in a precise quantification of smallness, and this is the second new feature of the present study. The pioneering work on the small-energy situation is due to Blythe (1969) and Ockendon & Spence (1969). Both papers deal with the one-dimensional unsteady piston problem, although Blythe makes brief mention of two-dimensional steady supersonic flow, but both have a very special definition of what they mean by small energies. If the amplitude of the gasdynamic disturbance is measured by the parameter ϵ, their small energies are also $O(\epsilon)$. We shall find that the comparable measure in axisymmetric flow is $O(\epsilon^4)$, which helps to emphasize the wide spectrum of small energies that then lie between 1 and ϵ^4, for example. The previous studies are confined explicitly or implicitly to $O(1)$ and/or $O(\epsilon)$ energies, whereas the present work explicitly includes the whole possible range of relaxing-mode energy. The special character of the $O(\epsilon^4)$ energy level is in no way diminished in its importance, but it appears likely that atmospheric relaxation energies could fall into the group above this extremely low value.

For these reasons we must begin with a careful examination of the basic linear-theory results, especially in so far as these begin to break down in their ability to predict wavelet position. The nature of the breakdown makes it possible to suggest the correct variable systems for the proper description of nonlinear wave behaviour. This analysis is concluded in § 3 for smooth, but otherwise general, body shapes. The present paper then devotes the remainder of its attention to the cone. This helps to simplify matters by diminishing the importance of the relaxation length as a parameter in the problem, and focuses interest on the role of the relaxing-mode energy level. We hope to report on extensions of this work to the case of bodies of finite length in the near future.

2. Conservation equations

The conservation equations for steady flow of an inviscid relaxing gas can be conveniently expressed in dimensionless form as

$$\mathbf{u}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\rho(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\nabla p = 0, \tag{2}$$

$$(\mathbf{u}\cdot\nabla)\,p+\rho a_f^2\,\nabla\cdot\mathbf{u}+\rho a_f^2\,\sigma(\mathbf{u}\cdot\nabla)\,q = 0, \tag{3}$$

$$\lambda(\mathbf{u}\cdot\nabla)\,q = M_{f0}(q_e-q), \tag{4}$$

where the variables are defined as follows: ρ is the density measured in units of the free-stream density ρ_0'; p is the pressure measured in units of $\rho_0' a_{f0}'^2$, where a_{f0}' is the free-stream frozen sound speed, so that $a_{f0}' a_f$ is the local frozen sound speed; $\mathbf{u}$ is the velocity vector divided by a_{f0}'; q is the non-equilibrium variable and q_e is its local equilibrium value, both based on the equilibrium free-stream value q_0'; ∇ is the dimensionless gradient operator with lengths measured in units of L', a typical body dimension. Of the remaining quantities

$$\lambda = \tau_0' U'/L' \tag{5}$$

defines a dimensionless 'relaxation length', with U' the local free-stream speed and τ_0' the (constant) relaxation time. We also require

$$M_{f0} = U'/a_{f0}', \tag{6}$$

the free-stream frozen Mach number, and

$$\sigma = -\left(\frac{\partial h}{\partial q}\right)_{p,\rho} \Big/ \rho \left(\frac{\partial h}{\partial \rho}\right)_{p,q}, \tag{7}$$

where h is the dimensionless enthalpy. Evidently σ is measured in units of $1/q_0'$.

It is convenient to use p and s, the entropy, to evaluate q_e and we observe that it is possible (see, for example, Clarke & McChesney 1976, p. 184) to write

$$\left(\frac{\partial q_e}{\partial s}\right)_{p,0} (\mathbf{u}\cdot\nabla)s \simeq \left\{\frac{(\partial h/\partial q)_{p,T,0}}{T_0(\partial h/\partial T)_{p,q,0}}\right\}\left(\frac{\lambda}{M_{f0}}\right)[(\mathbf{u}\cdot\nabla)q]^2 \tag{8}$$

to a first order of accuracy. A subscript zero indicates evaluation in the equilibrium free stream and T is the dimensionless absolute translational temperature. The reference quantities for h, s and T are not important since h, s and T always appear in the combinations found in (7) and (8).

It is also very important to write the conservation equations in a form which both identifies and exploits the existence of frozen, or high frequency, wavelets or characteristics. Since we are to study the steady-state axisymmetric case it is helpful to identity the velocity components u and v along the axial (x) and radial (r) directions respectively and then to define the flow deflexion angle θ, the modulus of the velocity vector V and the local frozen Mach angle μ_f as follows:

$$\theta = \tan^{-1}(v/u), \quad V^2 = u^2+v^2, \quad \mu_f = \sin^{-1}(a_f/V). \tag{9}$$

The required characteristic forms are then

$$\frac{D_f^{\mp} p}{Dx} \pm \rho \frac{V^2}{\beta_f}\frac{D_f^{\mp}\theta}{Dx} + \frac{\cos\mu_f \tan^2\mu_f}{\cos(\theta\pm\mu_f)}\left\{\rho V^2 \frac{\sin\theta}{r} + \frac{V}{a_f^2} W\right\} = 0, \tag{10}$$

where

$$\beta_f^2 = (V^2/a_f^2) - 1, \quad W = \rho a_f^2 \sigma (\mathbf{u}\cdot\nabla) q \tag{11}$$

and

$$\frac{D_f^{\mp}}{Dx} = \frac{\partial}{\partial x} + \tan(\theta\pm\mu_f)\frac{\partial}{\partial r}. \tag{12}$$

We observe that in this special axisymmetric situation ∇ has components $\partial/\partial x$ and $\partial/\partial r$, while

$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r}, \quad \nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv). \tag{13}$$

The ratio of the frozen sound speed a'_{f0} in the free stream to the corresponding equilibrium sound speed a'_{e0} is an important quantity, especially in so far as it is a measure of the fraction of the thermal energy of the gas that exists in the relaxing mode; it is related to variables already defined by

$$\sigma_0(\partial q_e/\partial p)_{s,0} = (a'_{f0}/a'_{e0})^2 - 1 = \hat{\delta}. \tag{14}$$

We shall find that $\hat{\delta}$ and λ, defined in (5), are two of the most important parameters in the problem. A third decisive parameter is the thickness ratio ϵ of the body, which appears in the equation for the body's meridian profile, namely

$$r = \epsilon R(x). \tag{15}$$

The cross-sectional area of the body is equal to $\epsilon^2\pi R^2(x)$ and since it occurs frequently it will be written henceforth as $\epsilon^2 S(x)$.

The task is now to solve (1)–(4) etc. for the variables p, ρ, u, v, q and q_e subject to the tangency condition

$$v(x, \epsilon R(x)) = \epsilon R'(x)\, u(x, \epsilon R(x)), \tag{16}$$

where $R' = dR/dx$, and also subject to the condition that the wave system is of the downstream-propagating type. Since $a'_{f0} > a'_{e0}$ ($\hat{\delta} > 0$) it is only necessary to ensure that $M_{f0} > 1$ in order to make the equilibrium Mach number

$$M_{e0} = U'/a'_{e0} \tag{17}$$

greater than one; the system of equations is then unequivocally hyperbolic and the wavelike character of the disturbance is guaranteed. Since $\epsilon \ll 1$ we shall seek approximate solutions along the lines of the now classical slender-body theory, most especially in so far as we set out to estimate the effects of relaxation on Whitham's (1950, 1952) far-field results.

It is first necessary to establish the solution of the linearized problem in the x, r co-ordinate system and since this was done some time ago (Clarke 1961), and recently repeated in the more up-to-date language of matched asymptotic expansions (Sinai 1975), there is no need to dwell on it here. Briefly, one finds that the perturbations of p, ρ, u, v, q and q_e are all of the same order in the mid-field limit ($\epsilon \to 0$; x, r fixed), and that a velocity potential exists for the first-order velocity perturbations. The solution for the potential involves an unknown source strength which can be found by applying condition (16) to the near-field version of the equations (obtained in the limit as $\epsilon \to 0$ with x and r/ϵ fixed) and then matching with the mid-field result. It transpires that all perturbations of p, etc., are $O(\epsilon^2)$ in the mid-field and that the source strength is proportional to $S'(x) = dS/dx$. Details are given in the next section.

3. Mid-field solution

The linear solution in x, r co-ordinates described in the previous section can be written in the form

$$\psi(x, r; \epsilon) = \psi^{(0)} + \epsilon^2 \psi^{(1)}(x, r)\{1 + O(\epsilon^2)\}, \tag{18}$$

where $\psi(x, r; \epsilon)$ is a column vector given by

$$\psi = \{p, \rho, u, v, q, q_e\}^{\mathrm{T}} \tag{19}$$

and $\psi^{(0)}$ is its undisturbed, or free-stream, value, i.e.

$$\psi^{(0)} = \{p_0'/\rho_0' a_{f0}'^2, 1, M_{f0}, 0, 1, 1\}^{\mathrm{T}}. \tag{20}$$

Then $\{p^{(1)}, \rho^{(1)}, u^{(1)}, v^{(1)}, q^{(1)}, q_e^{(1)}\}^{\mathrm{T}}$ will provide the first-order, linear, perturbation solution that we seek. For present purposes it will be sufficient to know the solution for $u^{(1)}$, which, if we assume that M_{f0} is neither transonic nor hypersonic in magnitude, can be written as

$$u^{(1)} = -\frac{M_{f0}}{2\pi}\int_0^{\xi-} S''(y)\left\{\frac{1}{2\pi i}\int_{Br} K_0(\beta_{f0} r z\chi)\, e^{(x-y)z}\,dz\right\}dy, \tag{21}$$

where S'' is the second derivative of S,

$$\xi = x - \beta_{f0} r, \tag{22}$$

$$\chi = \left\{\frac{\lambda z + (\beta_{e0}/\beta_{f0})^2}{\lambda z + 1}\right\}^{\frac{1}{2}}, \quad \operatorname{Re}\chi > 0, \tag{23}$$

$$\beta_{*0}^2 = M_{*0}^2 - 1, \quad * = e \quad \text{or} \quad f, \tag{24}$$

and K_0 is a zeroth-order modified Bessel function (defined, for example, in Abramowitz & Stegun 1965, p. 374). The integration contour Br for the complex variable z is parallel to the imaginary-z axis and lies to the right of the singularities of the integrand; the latter are at $z = -\beta_{e0}^2/\beta_{f0}^2\lambda,\ -1/\lambda$, 0 and $-\infty$.

We note the definitions and relations

$$(\beta_{e0}/\beta_{f0})^2 - 1 = b^2 - 1 = M_{f0}^2\hat{\delta}/\beta_{f0}^2 = \delta; \tag{25}$$

χ can thus be written as

$$\chi = \{1 + \delta/(1+\lambda z)\}^{\frac{1}{2}}. \tag{26}$$

We also observe that $\hat{\delta}$ and δ are of the same order of magnitude. It is clear that $u^{(1)}$ depends upon the variables x and r and the parameters λ and δ (or $\hat{\delta}$) as well as on the shape of the body via the function S''. Of course the Mach number M_{f0} also affects the value of $u^{(1)}$ but it has already been presumed to be of proper supersonic magnitude and therefore, unlike λ and δ, will not be permitted to take on extreme values of any kind.

It is now necessary to investigate the various ways in which the linearized solution, exemplified by (21), fails to predict the true wavelike character of the motion. The style and degree of this failure will enable new co-ordinates and variables to be defined which correct the deficiencies of the purely linear theory. As a first step in this process we must examine the behaviour in two critical regions; one is near to the linear wave head, where the failure of the linear theory to predict the existence of a shock wave is evident, and the second is far from the body (in a sense to be more carefully prescribed below), where the cumulative effects of convection and local sound-speed variation are to be anticipated.

The wave-head behaviour of $u^{(1)}$ is sought under the nominal condition $\xi/r \ll 1$, so that some element of 'distance from the body' can be included under this heading. The contour Br can be placed sufficiently far along the $+\operatorname{Re} z$ axis to ensure that $|\beta_{f0} r z\chi|$ is uniformly large; the asymptotic form of K_0 can then be used to show that

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{f0} r)^{\frac{1}{2}}}\exp\{-\beta_{f0} r\delta/2\lambda\}\left\{\frac{1}{2\pi}\int_0^{\xi}\frac{S''(y)}{(\xi-y)^{\frac{1}{2}}}dy\right\}, \tag{27}$$

provided that $\xi/\beta_{f0} r$, $\delta\xi/\lambda$ and $r\delta\xi/\lambda^2$ are all much less than unity.

There is an implication in (27) that $r\delta/\lambda$ is of order unity, even though δ/λ may be allowed to become arbitrarily small, for example. If $\delta/\lambda \to 0$ with r and ξ fixed (27) becomes

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{f0} r)^{\frac{1}{2}}} \left\{ \frac{1}{2\pi} \int_0^{\xi} \frac{S''(y)}{(\xi - y)^{\frac{1}{2}}} dy \right\} \tag{28}$$

under the sole condition that $\xi/\beta_{f0} r \ll 1$. Equation (28) is the familiar inert-gas result as discussed, for example, by Whitham (1974, p. 227) and the integral in brace brackets is often called the Whitham function.

When $r\delta/\lambda$ is of order unity (27) shows that relaxation effects on wave-head behaviour are confined to simple exponential decay with increasing $r\delta/\lambda$. Neither (27) nor (28) is valid in the limit of zero λ. Reasons for choosing either representation (27) or representation (28) are given at the end of this section.

It is best to begin the second task, namely examination of the behaviour of $u^{(1)}$ far from the body, by reverting to the complex-integral form of solution (21). This is

$$u^{(1)} = -M_{f0} \frac{1}{2\pi i} \int_{Br} \frac{\bar{S}''(z)}{2\pi} K_0(\beta_{f0} r z \chi)\, e^{xz}\, dz, \tag{29}$$

where $\bar{S}''(z)$ is the Laplace transform of S'', and with the nominal condition $r \gg 1$ the asymptotic form of K_0 can again be employed, but this time so as to write (29) in the form

$$u^{(1)} = -M_{f0} \frac{1}{2\pi i} \int_{Br} \frac{\bar{S}''(z)}{2\pi} \left(\frac{\pi}{2\beta_{f0} r z \chi} \right)^{\frac{1}{2}} \exp\left[xz - \beta_{f0} r z \chi\right] \left[1 + O\left(\frac{1}{\beta_{f0} r z \chi} \right) \right] dz. \tag{30}$$

For the present let us consider the first term in this representation of $u^{(1)}$. Writing it as

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{f0} r)^{\frac{1}{2}}} \frac{1}{2\pi i} \int_{Br} \left\{ z \frac{\bar{S}''(z)}{2\pi} \left(\frac{\pi}{z} \right)^{\frac{1}{2}} \right\} \left\{ \frac{1}{z\chi^{\frac{1}{2}}} \exp\left[xz - \beta_{f0} r z \chi\right] \right\} dz, \tag{31}$$

it is evidently possible to express $u^{(1)}$ as the convolution of the two groups of terms in brace brackets. Defining the Whitham function

$$\mathscr{W}(y) = \frac{1}{2\pi} \int_0^y \frac{S''(w)}{(y-w)^{\frac{1}{2}}} dw, \tag{32}$$

the first group in (31) is the transform of $\mathscr{W}'(y)$, since $\mathscr{W}(0)$ is zero, and (31) may therefore be written as

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{f0} r)^{\frac{1}{2}}} \int_0^{\xi -} \frac{d}{dy} \mathscr{W}(y) \frac{1}{2\pi i} \int_{Br} \frac{1}{z\chi^{\frac{1}{2}}} \exp\left[(x-y) z - \beta_{f0} r z \chi\right] dz\, dy. \tag{33}$$

The upper limit is $\xi -$ and not $x -$ as a result of the behaviour of the complex inner integral. It is now possible to integrate (33) by parts, integrating the $\mathscr{W}'(y)$ term first. This presents us with the necessity to differentiate the complex inner integral with respect to y; it is not possible to exchange the operations d/dy and $\int_{Br}$ since the resulting $\int_{Br}$ is not convergent but it is possible to reconcile

Br with an open loop contour $\mathscr{L}$ which surrounds the $-\text{Re}\, z$ axis, and which then permits interchanging d/dy and $\int_{\mathscr{L}}$. When all of this is done, (33) becomes

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{f0}r)^{\frac{1}{2}}}\exp\left[-\beta_{f0}r\delta/2\lambda\right]\mathscr{W}(\xi) - \frac{M_{f0}}{(2\beta_{f0}r)^{\frac{1}{2}}}\int_0^{\xi-}\mathscr{W}(y)\frac{1}{2\pi i}\int_{\mathscr{L}}\exp\left[\beta_{f0}rG(z;\mu)\right]\chi^{-\frac{1}{2}}\,dz\,dy, \quad (34)$$

where

$$G(z;\mu) = z[\mu-\chi], \quad \mu = (x-y)/\beta_{f0}r.$$

The wave-front result (27) is reproduced in the first term of (34) and evidently holds when ξ is small in some sense; the precise order of the required smallness is more easily comprehended from the direct approach which led to (27) and we are now concerned with $u^{(1)}$ when r is made large in a way which will become apparent. To this end we must examine

$$\mathscr{I} = \frac{1}{2\pi i}\int_{\mathscr{L}}\exp\left[\beta_{f0}rG(z;\mu)\right]\chi^{-\frac{1}{2}}\,dz. \quad (35)$$

Of course integrals like $\mathscr{I}$ have been studied before in connexion with planar-supersonic and piston-problem flows (Clarke 1960, 1965) and in quite some generality of circumstances by Whitham (1974). However, all of these analyses contain an implicit restriction to values of δ (in the present terminology) which are of roughly unit magnitude. We are especially interested in a whole spectrum of small values of δ and a careful re-assessment of the behaviour of $\mathscr{I}$ is called for.

Rewriting $\mathscr{I}$ in the form

$$\mathscr{I} = \frac{1}{2\pi i\lambda}\int_{\mathscr{L}}\exp\left[\bar{r}\bar{G}(w;\mu)\right]\bar{\chi}^{-\frac{1}{2}}\,dw, \quad (36)$$

where

$$\bar{r} = \beta_{f0}r/\lambda, \quad \bar{G}(w;\mu) = w[\mu-\bar{\chi}], \quad \bar{\chi} = \{1+\delta/(1+w)\}^{\frac{1}{2}},$$

we shall employ the method of steepest descents to evaluate the integral on the nominal presumption that $\bar{r} \gg 1$. The function $\bar{G}$ has saddle points at those values of w, called w_0, for which $\partial\bar{G}/\partial w$ vanishes; evidently

$$\mu - \bar{\chi}(w_0) - w_0\,\bar{\chi}'(w_0) = 0. \quad (37)$$

Any w_0 on the sheet for which $\text{Re}\,\bar{\chi} > 0$ must be real, so that $\bar{G}(w_0;\mu)$ is real too. Observing from (37) that w_0 is a function of μ it follows that

$$d\bar{G}(w_0;\mu)/d\mu = w_0.$$

The stationary value of $\bar{G}$ at $w_0 = 0$ is a maximum, with $\bar{G}(0;b) = 0$; the value b of μ at $w_0 = 0$ follows from the definitions in (25) and (36). Furthermore, since $1 < \mu < x/\beta_{f0}r$, any relevant w_0 lies in $\infty > w_0 > -1$ and $\bar{G}(w_0,\mu) < 0$ everywhere except at $w_0 = 0$.

The steepest path $\mathscr{S}$ through any saddle point w_0 can be shown to be $\mathscr{L}$-like and in fact to be locally parabolic in shape with vertex at w_0. The integral $\mathscr{I}$ will be proportional to $\exp\left[\bar{r}\bar{G}(w_0;\mu)\right]$ and hence for any value of μ other than b it must become small when $\bar{r} \gg 1$. This latter statement requires qualification since it must depend to some degree on the rate at which $\bar{G}(w_0;\mu)$ diminishes from zero as μ moves away from b. Since this special value of μ is clearly significant we are led to rewrite $\mathscr{I}$ as

$$\mathscr{I} = \frac{1}{2\pi i\lambda}\int_{\mathscr{S}_0} \exp(\mu_e \bar{r} w)\exp[\bar{r}\bar{G}(w;b)]\,\bar{\chi}^{-\frac{1}{2}}\,dw, \tag{38}$$

where $\mathscr{S}_0$ is the steepest path through the saddle point of $\bar{G}(w;b)$, namely the origin $w = 0$, and μ_e is given by

$$\mu_e = (\xi_e - y)/\beta_{f0} r, \quad \xi_e = x - \beta_{e0} r. \tag{39}$$

The real variable t on $\mathscr{S}_0$ is usually defined by

$$-t^2 = \bar{G}(w;b) - \bar{G}(0;b) = \tfrac{1}{2}w^2\bar{G}''(0;b) + \tfrac{1}{6}w^3\bar{G}'''(0;b) + \dots,$$

and the definition of $\bar{G}$ and $\bar{\chi}$ in (36) shows that

$$\bar{G}^{(n)}(0;b) = -n\bar{\chi}^{(n-1)}(0), \quad n = 2, 3, 4, \dots .$$

It is evident that $\bar{\chi}^{(n-1)}(0)$ is proportional to δ for any $n \geqslant 2$ and since δ must be admitted to be a very small number in many practical situations it is necessary to redefine the steepest-path variable as follows:

$$-t^2\delta = \bar{G}(w;b) = -w^2\bar{\chi}'(0) - \tfrac{1}{2}w^3\bar{\chi}''(0) - \dots . \tag{40}$$

Evaluating $\bar{\chi}'(0)$, etc., we find that

$$-t^2 = \frac{w^2}{2b} - \left(\frac{3b^2+1}{8b^3}\right)w^3 + O(w^4). \tag{41}$$

Inversion of this series gives

$$w = e^{\pm\frac{1}{2}i\pi}(2b)^{\frac{1}{2}}t - \left(\frac{3b^2+1}{4b}\right)t^2 + O(t^3), \tag{42}$$

where the upper (lower) sign applies on the upper (lower) half of $\mathscr{S}_0$. It can be confirmed that dw/dt is just the t derivative of (42) and it is significant that the coefficients of t in (42) are of order unity. Relations (40) and (42) can now be introduced into (38) and an asymptotic estimate of the behaviour of $\mathscr{I}$ can be sought under the correct large $-\bar{r}$ condition, which, it is now clear, must be

$$\bar{r}\delta = \beta_{f0} r\delta/\lambda \gg 1. \tag{43}$$

It can also be seen that the asymptotic development of (38) is only valid if μ_e is restricted in size. If μ_e departs too much from zero the saddle point must move from the origin; use of (40) and (42) in the index of the exponentials in (38) demonstrates that for the origin to be the correct saddle point $|\mu_e|\bar{r}(3b^2+1)/4b$ must be much less than $\bar{r}\delta$, and this condition is most concisely expressed in the form

$$|\mu_e|/\delta \ll 1, \tag{44}$$

since b is very nearly equal to unity.

Now (39) defines μ_e and since (34) requires $0 \leqslant y < \xi$ it follows that

$$\xi_e/\beta_{f0} r \geqslant \mu_e > -(b-1) = -\delta/(b+1).$$

Evidently $|\mu_e|/\delta$ is not small for y near to ξ, and (44) is violated in these regions. However, we note two facts: first, evaluating $\mathscr{I}$ from (38), (40), (42), etc., gives

$$\mathscr{I} \sim \{2\pi\bar{r}\delta\lambda^2\}^{-\frac{1}{2}} \exp[-\mu_e^2 \bar{r}b/2\delta], \tag{45}$$

and second, $|\mu_e|/\delta$ is at most about $\frac{1}{2}$ for y near to ξ. Since (43) requires $\bar{r}\delta$ to be arbitrarily large, $\mathscr{I}$ is exponentially small for y near to ξ and this behaviour is at least qualitatively correct, as can be seen by consulting the paragraph prior to (38). Thus (45) is a proper asymptotic form for $\mathscr{I}$ throughout the range of y, provided that the restriction (43) is met and provided that it is supplemented by the correct interpretation of (44), which is clearly

$$|\xi_e|/\beta_{f0}\, r\delta \ll 1. \tag{46}$$

It can also now be shown that the term neglected in going from (30) to (31) is, by virtue of the behaviour of $\mathscr{I}$, negligible if $|\xi_e|/\beta_{f0}\, r \ll 1$; this condition is implicit in (46) in the present circumstances. Combination of (45) and (34) now gives

$$u^{(1)} \sim -\frac{M_{f0}}{(2\beta_{e0}\, r)^{\frac{1}{2}}} \int_0^\infty \mathscr{W}(y) \left\{ \frac{\exp[-(\xi_e - y)^2/2\beta_{e0}\, r\lambda\delta b^{-2}]}{(2\pi\beta_{e0}\, r\,\lambda\delta b^{-2})^{\frac{1}{2}}} \right\} dy. \tag{47}$$

The upper limit has been written as ∞ since this involves neglect of only exponentially small terms, which is implict in the evaluation of $\mathscr{I}$ anyway. We reiterate that the errors in (47) are $O(\lambda/\beta_{f0}\, r\delta)$ and $O(|\xi_e|/\beta_{f0}\, r\delta)$.

Result (47) is completely analogous to the large-time result for the piston problem (Clarke 1965), the interesting fact emerging from (47) that it is the Whitham function $\mathscr{W}(y)$ which acts here as the 'piston-velocity input' to the perturbation field. It is readily seen that $u^{(1)}$ tends to the classical equilibrium far-field result for slender-body theory in the limit of zero λ. We remark that the limit of zero δ is properly derived from (28) and not from the present result, which requires $r\delta/\lambda \gg 1$.

The interpretation of (47) as a solution of the initial-value problem for a one-dimensional diffusion equation is well known and we observe that the appropriate diffusivity is $\beta_{e0}\,\lambda\delta b^{-2}$. This can be rewritten as $M_{e0}^4\, \nu_v/\beta_{e0}$, where ν_v is the equivalent bulk (kinematic) viscosity measured, in the present example, in units of $U'L'$. Since β_{e0} and b are of roughly unit magnitude the diffusivity can be taken to be of order $\lambda\delta$.

Finally in this section, we reiterate that we have investigated the behaviour of the mid-field solution in two regions which will prove to be important because of their bearing on the nonlinear wave system. This has been done, first, on the presumption that δ and λ are two parameters which remain fixed under the mid-field limiting process, which is $\epsilon \to 0$ with r, x fixed and which leads to the basic form of the solution in (21) or (29). Second, having derived these solutions, we should like to propose that both δ and λ *may* have independently extreme values; for example (28) provides the requisite form of $u^{(1)}$ when $\delta/\lambda \to 0$ provided that ξ/r is suitably small, and (47) gives the correct equilibrium-state form of $u^{(1)}$ when $\lambda \to 0$ *provided that $\xi_e/r\delta$ is properly small*. The last proviso helps us to make the important point that it is vital to ensure that δ is *not* allowed to become arbitrarily small, like ϵ^n, $n > 0$, say, without the most careful attention to the details of mid-field behaviour. This is most immediately important in relation to results (27) and (28) when it is recognized that $\exp[-\beta_{f0} r\delta/2\lambda] = 1 + O(\epsilon^2)$ for any $\delta/\lambda \leqslant \mathrm{ord}\, \epsilon^2$ in the mid-field, where r is fixed, and that the resulting '$\epsilon^2 O(\epsilon^2)$' terms in the representation of ψ [see (18)] belong in the next and higher-order terms like $\epsilon^4\psi^{(2)}$; we are essentially concerned with $\epsilon^2\psi^{(1)}$ only. The implication is that when $\delta/\lambda \leqslant \mathrm{ord}\, \epsilon^2$ relaxation effects are properly of second order in the mid-field and that (28) must then be used to describe wave-head conditions in this part of the flow.

The role of the far-field solution (47) will emerge as the analysis proceeds. Clearly condition (43) can always be met for small enough values of λ and we must assume at this stage that (47) correctly represents $u^{(1)}$ and does not, illegally from the point of view of the mid-field solution, contain any elements that should be consigned to $\epsilon^4\psi^{(2)}$ or higher terms. Some comments on this assumption are made at the end of the next section.

4. The cone

Although the mid-field results derived in the previous section are valid for general smooth body meridian profiles it is both instructive and intrinsically interesting to examine the relatively simple case of the right circular cone. In that case S'' is equal to 2π everywhere and the Whitham function is simply

$$\mathscr{W}(y) = 2y^{\frac{1}{2}}.$$

Since there is no typical body length L' in this case it is best to choose L' to be $U'\tau_0'$, so that we henceforth write $\lambda = 1$ in most of this section. This enables us to concentrate on the effects of the relaxing-mode energy content, as expressed by the quantity δ, or $\hat{\delta}$, without the additional complication of relaxation-length to body-length variations, although we do find it expedient to re-introduce λ explicitly towards the end of the present analysis. It should be remarked that when there is no L', as is the case with the cone, the definition of a 'thickness ratio' ϵ must be modified to mean the semi-nose angle of the conical body; (15) is just $r = \epsilon x$ in these circumstances.

4.1. *Frozen wave head; intermediate energy*

When $\mathrm{ord}\, 1 \geqslant \delta > \mathrm{ord}\, \epsilon^2$ we find from (18)–(20) and (27) that, provided

$$\xi/r \ll 1, \quad \xi \ll 1$$

and $r\delta$ is $O(1)$,

$$u = M_{f0} - \epsilon^2 2M_{f0} \exp[-\tfrac{1}{2}\beta_{f0} r\delta]\,(\xi/2\beta_{f0} r)^{\frac{1}{2}} \tag{48}$$

near the frozen wave head. It is now necessary to examine the way in which this result fails to predict u correctly in these regions. We remark that $\xi \ll 1$ implies $\delta\xi \ll 1$. It is legitimate to take $r\delta$ as a quantity of order unity here for the reasons explained at the end of the previous section which relate to the higher-order terms in the mid-field expansion; it also enables us to deal conveniently with the order range of δ quoted above.

The high-frequency wavelets are located on surfaces given [see (12) and (9)] by

$$dr/dx = \tan(\theta \pm \mu_f) = (a_f^2 - v^2)\,[\pm a_f(u^2 - a_f^2)^{\frac{1}{2}} - uv]^{-1}, \tag{49a}$$

and the mid-field estimates show that

$$\frac{dr}{dx} = \pm\frac{1}{\beta_{f0}}\left\{1 + \epsilon^2\left[\frac{M_{f0}^2}{\beta_{f0}^2}a_f^{(1)} - \frac{M_{f0}}{\beta_{f0}^2}u^{(1)} \pm \frac{M_{f0}}{\beta_{f0}}v^{(1)}\right] + o(\epsilon^2)\right\}. \tag{49b}$$

Observe that, consistent with (18), etc., $a_f = 1 + \epsilon^2 a_f^{(1)}$. The mid-field solution requires

$$v^{(1)} = -\beta_{f0}u^{(1)}, \quad p^{(1)} = -M_{f0}u^{(1)} \tag{50}$$

and it can be shown that

$$a_f^{(1)} = (\Gamma_{f0} - 1)\,p^{(1)} = -(\Gamma_{f0} - 1)\,M_{f0}u^{(1)}, \tag{51}$$

where

$$\Gamma_f = \frac{1}{a_f}\left(\frac{\partial(\rho a_f)}{\partial \rho}\right)_{s,q}. \tag{52}$$

We remark that the derivation of (51) is not altogether a trivial matter but, for the sake of brevity, we shall confine ourselves to the remark that it depends upon finding that the variations in a_f with entropy s [note (8)] and with the non-equilibrium variable q are negligible compared with the variations due to changes in pressure; it is therefore necessary to have $(\partial q_e/\partial p)_s$ and $(\partial q_e/\partial s)_p$ of order unity in the limit as $\epsilon \to 0$.

Equations (49b), (50) and (51) give the approximate form of the shape of the high-frequency wavelets for outward-propagating waves, namely

$$\frac{dr}{dx} = \frac{1}{\beta_{f0}}\left\{1 - \epsilon^2\frac{M_{f0}^3}{\beta_{f0}^2}\Gamma_{f0}u^{(1)}\right\},$$

and this can be rewritten in the form

$$\frac{d\xi}{dr} = \epsilon^2\frac{M_{f0}^3}{\beta_{f0}}\Gamma_{f0}u^{(1)} = -\epsilon^2\frac{2M_{f0}^4}{\beta_{f0}}\exp\left(-\tfrac{1}{2}\beta_{f0}r\delta\right)\left(\frac{\xi}{2\beta_{f0}r}\right)^{\frac{1}{2}}.$$

We can therefore estimate the wavelet shape to be roughly

$$\xi^{\frac{1}{2}} \simeq -\epsilon^2\delta^{-\frac{1}{2}}C\,\mathrm{erf}\,(\tfrac{1}{2}\beta_{f0}r\delta)^{\frac{1}{2}} + x_0^{\frac{1}{2}},$$

where $\xi = x_0$ when $r = 0$, and C is a constant of order unity. But the linear theory makes $d\xi/dr = 0$, or $\xi^{\frac{1}{2}} = x_0^{\frac{1}{2}}$, so there is an error in the frozen-wavelet location of roughly $\xi^{\frac{1}{2}} - x_0^{\frac{1}{2}} \approx -\epsilon^2 C/\delta^{\frac{1}{2}}$. When this position error in ξ is as large as ξ itself it is clear that the linear theory has broken down, since it entirely fails to get the wavelet positions anywhere near correct, and a revised co-ordinate system must be employed; since breakdown occurs when $\xi = O(\epsilon^4/\delta)$ with $r\delta$ of order unity, the new co-ordinates should evidently be

$$\Xi = \xi\delta/\epsilon^4, \quad R = r\delta \tag{53}$$

and the velocity component u should be written as

$$u = M_{f0} = \epsilon^4 U^{(1)}(\Xi, R). \tag{54}$$

The momentum equations (2) now show that (53) and (54) require

$$v = -\epsilon^4\beta_{f0}\,U^{(1)}, \quad p = (p_0'/\rho_0' a_{f0}'^2) - \epsilon^4 M_{f0}\,U^{(1)}. \tag{55}$$

The equilibrium value q_e of q will be given by

$$q_e = 1+\epsilon^4(\partial q_e/\partial p)_{s,0}P^{(1)}, \quad P^{(1)} = -M_{f0}\,U^{(1)}, \tag{56}$$

and it then follows that q must be written as

$$q = 1+\epsilon^8\delta^{-1}Q^{(1)}(\Xi, R), \tag{57}$$

with $Q^{(1)}$ found from the relation

$$Q^{(1)}_{\Xi} = (\partial q_e/\partial p)_{s,0}P^{(1)}. \tag{58}$$

Equation (58) is valid only when $\delta > \operatorname{ord}\epsilon^4$ and we also observe that ξ will be only $o(1)$ in the region of linear-theory breakdown under this condition. Thus (48) will be a proper estimate of the behaviour of $u^{(1)}$ only when δ is large enough to meet this inequality; but we have already insisted that $\delta > \operatorname{ord}\epsilon^2$, so δ must meet the less restrictive condition here. Writing (48) in terms of Ξ and R and using the limiting process $\epsilon \to 0$ with these co-ordinates fixed makes

$$u \sim M_{f0} - \epsilon^4\{2M_{f0}\exp(-\tfrac{1}{2}\beta_{f0}R)(\xi\delta/2\beta_{f0}R\epsilon^{-4})^{\frac{1}{2}}\} \tag{59}$$

in the ξ, R co-ordinate system.

We must ensure that $U^{(1)}(\Xi, R)$ in (54) is such as to make that estimate of u behave like (59) in the relevant mid-field limit, which in view of (48) and its validity conditions must be $\epsilon \to 0$ with ξ and R (or $r\delta$) fixed.

The upper-sign version of (10) can be employed to find the equation for $U^{(1)}$. To do this it is necessary to use (9), (11), (14) and (49*a*), as well as all of the results (53)–(68). After some elementary manipulations the nonlinear equation

$$2M_{f0}^3\beta_{f0}^{-1}\Gamma_{f0}\,U^{(1)}U^{(1)}_{\Xi} + 2U^{(1)}_R + R^{-1}U^{(1)} + \beta_{f0}\,U^{(1)} = 0 \tag{60}$$

is found and the parametric solution for $U^{(1)}$ is

$$U^{(1)} = H(\alpha)\exp(-\tfrac{1}{2}\beta_{f0}R)\,R^{-\frac{1}{2}}. \tag{61}$$

The parameter α is to be found from

$$\Xi = G(\alpha) + \frac{M_{f0}^3}{\beta_{f0}}\Gamma_{f0}H(\alpha)\left(\frac{2\pi}{\beta_{f0}}\right)^{\frac{1}{2}}\operatorname{erf}\left(\frac{\beta_{f0}R}{2}\right)^{\frac{1}{2}} \tag{62}$$

and the functions $G(\alpha)$ and $H(\alpha)$ must be chosen so as to satisfy the matching requirement set out in (59).

Equation (62) gives $\alpha(\Xi, R)$ as an implicit function and matching requires that we should now write Ξ as $\xi\delta/\epsilon^4$ and let $\epsilon \to 0$ with ξ and R fixed. Comparing (59) and (61), we evidently require $H(\alpha)$ to behave like $-2M_{f0}(\xi\delta/2\beta_{f0}\epsilon^4)^{\frac{1}{2}}$ under these conditions; (62) shows that under similar conditions

$$\xi = \epsilon^4\delta^{-1}G(\alpha(\Xi\to\infty, R)) + O(\epsilon^2\xi^{\frac{1}{2}}\delta^{-\frac{1}{2}}),$$

so that $G(\alpha(\Xi\to\infty, R))$ must be $\alpha\delta/\epsilon^4$. The latter condition is most simply met by making

$$G(\alpha) = \alpha\delta/\epsilon^4. \tag{63}$$

Then $\xi = \alpha(1+O(\epsilon^6/\delta^{\frac{3}{2}}))$ in the matching region and

$$H(\alpha) = -2M_{f0}(\alpha\delta/2\beta_{f0}\epsilon^4)^{\frac{1}{2}}. \tag{64}$$

The nonlinear solution for u can now be written as

$$u = M_{f0} - \epsilon^2 2M_{f0}(\alpha/2\beta_{f0}r)^{\frac{1}{2}}\exp(-\tfrac{1}{2}\beta_{f0}r\delta), \tag{65}$$

where

$$\xi = \alpha - \epsilon^2(2M_{f0}^4\Gamma_{f0}/\beta_{f0}^2)(\pi\alpha/\delta)^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\beta_{f0}r\delta)^{\frac{1}{2}}, \tag{66}$$

and we reiterate that it is valid when $\operatorname{ord} 1 \geqslant \delta > \operatorname{ord}\epsilon^2$. The range of relaxing-mode energies identified by this limitation can be called the *intermediate energy* range.

When the wavelet shape can be represented by the general formula

$$\xi = \alpha - f(\alpha)\,g(r) \tag{67}$$

the rule for fitting a weak shock wave into the field can be written (Whitham 1974, p. 334) as

$$\xi_s = \alpha_1 - f(\alpha_1)\,g(r_s) = \alpha_2 - f(\alpha_2)\,g(r_s), \tag{68}$$

$$2\int_{\alpha_1}^{\alpha_2} f(y)\,dy = (\alpha_2-\alpha_1)\,[f(\alpha_2)+f(\alpha_1)], \tag{69}$$

where a subscript s denotes a value on the shock surface and α_1 and α_2 are wavelets running into the shock wave from upstream and downstream positions respectively. We are of course only concerned with a head, or bow, shock in the present situation, so that (69) simplifies to

$$2\int_0^{\alpha_2} f(y)\,dy = f^2(\alpha_2)\,g(r_s). \tag{70}$$

Comparing (66) and (67) it follows that α_2 and r_s are related by

$$\alpha_2 = \frac{9}{16}\left[\epsilon^2\frac{2M_{f0}^4}{\beta_{f0}^2}\Gamma_{f0}\right]^2\frac{\pi}{\delta}\,[\operatorname{erf}(\tfrac{1}{2}\beta_{f0}r_s\delta)^{\frac{1}{2}}]^2 \tag{71}$$

and that the shock wave therefore lies where (see figure 1)

$$x_s = \beta_{f0}r_s - \frac{\epsilon^4}{\delta}\frac{3\pi}{4}\frac{M_{f0}^8}{\beta_{f0}^4}\Gamma_{f0}^2\,[\operatorname{erf}(\tfrac{1}{2}\beta_{f0}r_s\delta)^{\frac{1}{2}}]^2. \tag{72}$$

Since this shock wave must be a Rankine–Hugoniot discontinuity across which q is frozen at its free-stream value of unity it is necessary to ensure that $Q^{(1)}$ in (57) is zero just behind the surface given by (72). This condition must therefore be used when $Q^{(1)}$ is found from (58). The first-order estimates of u, v, p, etc., can be found from (55) and (56) in conjunction with (65); values immediately behind the shock must make use of (71) to identify this special location.

Equation (72) shows that near the nose of the body, where $r_s \to 0$, the shock is a right circular cone whose shape is exactly that of a fully frozen-flow wave (e.g. Whitham 1974, p. 335).

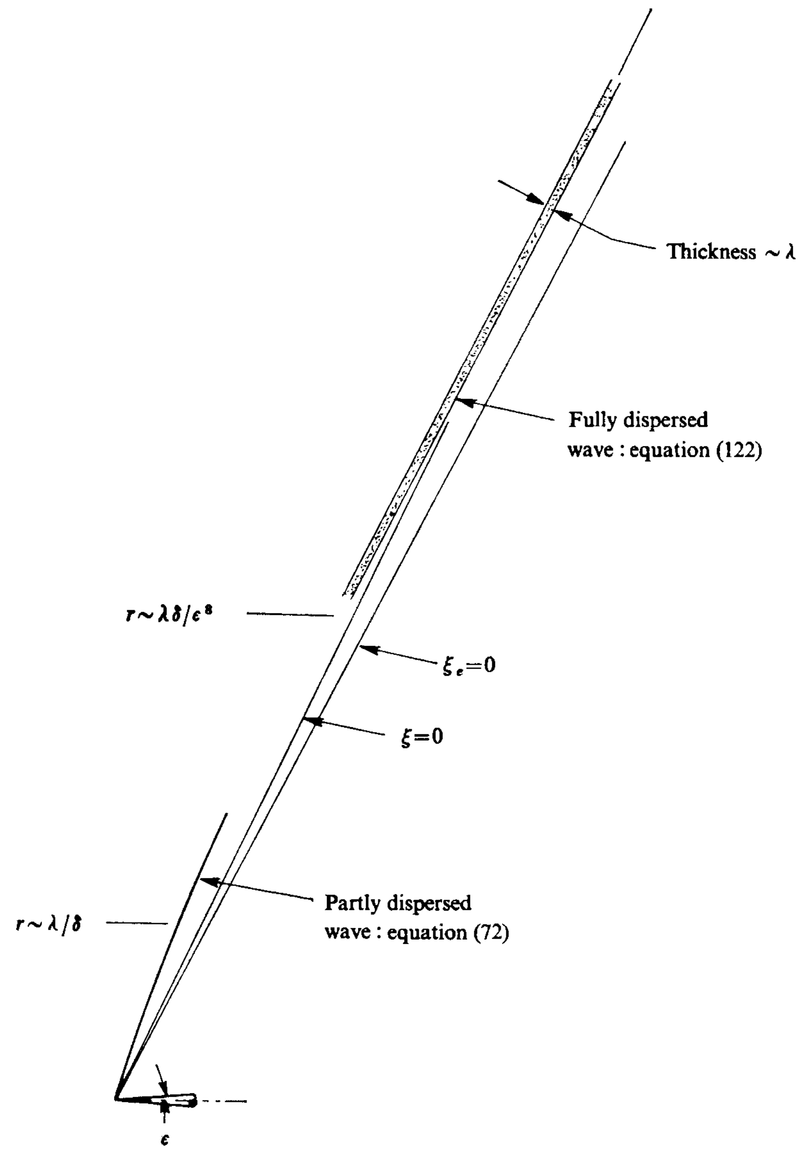

FIGURE 1. A sketch of the shock-wave system for ord $1 \geqslant \delta >$ ord ϵ^4, namely intermediate and small-intermediate relaxing-mode energy levels (see §§ 4.1 and 4.2). The partly dispersed wave has had its axial distance from the frozen free-stream Mach cone $\xi = 0$ greatly exaggerated so as to convey some impression of its shape; the true distance should be about $\frac{1}{50}$ of that depicted here when $\epsilon \sim 10^{-1}$. If $r \sim \lambda/\delta$ is one unit of radius the fully dispersed shock should appear in its substantially evolved form *not* at about 3 units of r as depicted, but at about $3(\delta/\epsilon^4)^2$ units; e.g. if $\delta = \epsilon^3$ and $\epsilon \sim 10^{-1}$ this is about 300 units. The axial location of the fully dispersed wave is uncertain within a distance of order λ.

4.2. *Frozen wave head; small intermediate energy*

If $\delta \leqslant \operatorname{ord} \epsilon^2$ we must use (28) to give

$$u = M_{f0} - \epsilon^2 2M_{f0}(\xi/2\beta_{f0} r)^{\frac{1}{2}} \tag{73}$$

provided that $\xi/r \ll 1$.

Following the same arguments regarding wavelet location as were used in § 4.1 one readily finds that linear theory breaks down when $\xi \approx \epsilon^4 r$ in the present case. Thus the requisite rescaling of co-ordinates is revealed for the ratio ξ/r, but one must decide how to arrange for the stretching of ξ and r separately. We note first that the nonlinear wave-head value for u must be precisely that in (54) and it soon follows from the conservation equations that all of the results in (55)–(58) are correct, provided that Ξ and R in $U^{(1)}(\Xi, R)$ are exactly as in (53), and provided that $\delta > \operatorname{ord} \epsilon^4$. The results in (60)–(62) are likewise valid here and the only difference arises from the necessity of matching these results with (73) rather than (48). Writing (73) in terms of Ξ and R and using the limit process $\epsilon \to 0$ with Ξ and R fixed makes

$$u - M_{f0} \sim -\epsilon^4 2M_{f0}(\Xi/2\beta_{f0} R)^{\frac{1}{2}} = -\epsilon^4 2M_{f0}(\xi/2\beta_{f0} r\epsilon^4)^{\frac{1}{2}},$$

where the last form follows when the result is expressed in mid-field co-ordinates ξ and r (N.B. *not* ξ and R as in § 4.1). It quickly follows that matching makes H and G exactly as in (63) and (64), and all the subsequent results of § 4.1 are valid in the present case, for which $\operatorname{ord} \epsilon^2 \geqslant \delta > \operatorname{ord} \epsilon^4$.

The fact that the present far-field behaviour is identical with the previous case, for which $\operatorname{ord} 1 \geqslant \delta > \operatorname{ord} \epsilon^2$, is not entirely trivial since relaxation effects are negligible in the present mid-field and are not negligible in the previous, larger-energy, situation.

In the whole energy spectrum $\operatorname{ord} 1 \geqslant \delta > \operatorname{ord} \epsilon^4$ we can now say that as r_s increases the shock bends back towards a free-stream frozen-Mach-cone shape and its strength diminishes in proportion to $\bar{R}^{-\frac{1}{2}} \operatorname{erf} \bar{R}^{\frac{1}{2}} \exp(-\bar{R})$, where $\bar{R} = \frac{1}{2}\beta_{f0} r\delta$. Since $\lambda = 1$ in the present section, r is measured in units of the relaxation length but it is important to observe that significant relaxation attenuation, which is present when $\bar{R} \geqslant 1$, say, may well appear only after many multiples of this length since the smallness of δ is only restricted by $\delta > \operatorname{ord} \epsilon^4$. For example, if $\delta = \epsilon^3$ and $\epsilon = 10^{-1}$, r must be of the order of 10^3 relaxation lengths for noticeable shock attenuation to appear (see figure 1).

A very similar result to (72) has been given by Chou & Chu (1971) for the nose shock shape on a general axisymmetric body when, as they put it, "the decay length is of the same order as a typical length scale of the projectile." Translated into our terminology this limitation is equivalent to taking $\delta = 1$, so that (72) extends Chou & Chu's result in an important way at the expense, so far, of restriction to a conical body. We comment that their method (of characteristic parameters) is very different from ours and cannot be extended to encompass other far-field behaviour, as can the present technique.

4.3. *Frozen wave head; small energy*

In the special situation in which $\delta = \operatorname{ord}\epsilon^4$ it is necessary to use (73) for the mid-field when $\xi/r \ll 1$. As in §4.2, the nonlinear far field must exist when $\xi \approx \epsilon^4 r$ and Ξ and R prove to be the correct co-ordinates in this domain. It is worth noting that, with $\delta = A\epsilon^4$, Ξ is equal to $A\xi$ and R is equal to $Ar\epsilon^4$; A is of course an $O(1)$ constant. Results (54)–(56) also apply in this case, as indeed does (57), which is repeated here in its special form

$$q = 1+\epsilon^4 A^{-1} Q^{(1)}(\Xi, R) \tag{74}$$

to draw attention to the fact that variations in q are now of the same order as those in q_e [see (56)]. Unlike the two previous cases the wave-head region is *not* nearly frozen and this is further exemplified by the fact that (58) must be replaced by

$$Q^{(1)}_{\Xi} = \left(\frac{\partial q_e}{\partial p}\right)_{s,0} P^{(1)} - \frac{1}{A} Q^{(1)}, \quad A = \delta/\epsilon^4. \tag{75}$$

It follows that the 'source' term W in (10), which is proportional to $Q^{(1)}_{\Xi}$, is now quite different from its form in the intermediate and small intermediate energy situations. The equation satisfied by $U^{(1)}$ is no longer (60) but has the less tractable form

$$\left(\frac{\partial}{\partial \Xi} + \frac{1}{A}\right)\left\{2U^{(1)}_R + \frac{1}{R} U^{(1)} + 2\frac{M^3_{f0}}{\beta_{f0}} \Gamma_{f0} U^{(1)} U^{(1)}_{\Xi}\right\} + \beta_{f0} U^{(1)}_{\Xi} = 0, \tag{76}$$

Equation (76) describes a cylindrically symmetric type of flow which is exactly analogous to the small-energy piston-problem motion discussed by Blythe (1969) and Ockendon & Spence (1969). Blythe includes steady two-dimensional supersonic flow in his analysis, which can therefore be used to describe the flow past a wedge. If the wedge is of thickness ratio ϵ Blythe's small energy level requires δ to be $O(\epsilon)$, so the contrast with the present small energy level, which makes δ equal to $O(\epsilon^4)$ for a *cone* of thickness ratio ϵ, is significant.

There are apparently no simple analytical solutions of (76), as there are of (60), but some useful pointers to the general nature of its solutions can be found by studying its characteristic form. In this we follow Blythe's approach to the piston problem, but although the technique is the same, the changed geometry here makes the results somewhat different.

First, defining $w = w(\mathscr{X}, \mathscr{Y})$ via

$$w = (2^{\frac{1}{2}} M^3_{f0}/\beta^{\frac{3}{2}}_{f0} A)\, \Gamma_{f0} R^{\frac{1}{2}} U^{(1)}, \quad \mathscr{X} = \Xi/A, \quad \mathscr{Y} = (\tfrac{1}{2}\beta_{f0} R)^{\frac{1}{2}}, \tag{77}$$

it transpires that w satisfies a parameter-free equation, namely

$$(w_{\mathscr{Y}} + 2ww_{\mathscr{X}})_{\mathscr{X}} + w_{\mathscr{Y}} + (2w+\mathscr{Y})\, w_{\mathscr{X}} = 0, \tag{78}$$

whose characteristics are $\mathscr{Y}$ = constant and α = constant, where α is defined by

$$(\partial\mathscr{X}/\partial\mathscr{Y})_\alpha = 2w. \tag{79}$$

Transforming to independent variables α and $\mathscr{Y}$ in place of $\mathscr{X}$ and $\mathscr{Y}$ and using (78) and (79) gives the following equation for $\mathscr{X} = \mathscr{X}(\alpha, \mathscr{Y})$:

$$\mathscr{X}_{\alpha\mathscr{Y}\mathscr{Y}} + \mathscr{X}_{\alpha}\mathscr{X}_{\mathscr{Y}\mathscr{Y}} + 2\mathscr{Y}\mathscr{X}_{\alpha\mathscr{Y}} = 0. \tag{80}$$

Since

$$u = M_{f0} + \epsilon^4 U^{(1)}(\Xi, R)$$

must match with the mid-field result for ord $\epsilon^2 \geqslant \delta$, namely (73), it is necessary to make

$$\{R^{\frac{1}{2}} U^{(1)}(\Xi, R)\}_{R\to 0} \to -2M_{f0}(\Xi/2\beta_{f0})^{\frac{1}{2}}, \tag{81}$$

so that (77) and (79) give

$$w(\mathscr{X}, 0) = -\tfrac{1}{2}K\mathscr{X}^{\frac{1}{2}}, \quad K = 4M_{f0}^4\Gamma_{f0}/\beta_{f0}^2 A^{\frac{1}{2}}, \tag{82}$$

when $\mathscr{X} \geqslant 0$. Since $w(\mathscr{X}, 0) = 0$ for $\mathscr{X} < 0$, w is continuous on $\mathscr{Y} = 0$; the simplest solution of (79), which identifies α with $\mathscr{X}$ on $\mathscr{Y} = 0$, is therefore

$$\mathscr{X} = \alpha + 2\int_0^{\mathscr{Y}} w(\mathscr{X}, s)\, ds. \tag{83}$$

Hence (82) translates into the condition

$$\mathscr{X}_{\mathscr{Y}}(\alpha, 0) = \begin{Bmatrix} -K\alpha^{\frac{1}{2}}, & \alpha \geqslant 0, \\ 0, & \alpha < 0, \end{Bmatrix} \tag{84}$$

on $\mathscr{X}$, supplemented by

$$\mathscr{X}(\alpha \leqslant 0, \mathscr{Y}) = \alpha, \quad \mathscr{X}(\alpha, 0) = \alpha. \tag{85), (86}$$

Thus $\alpha = 0$ is the wave head, downstream of which $\mathscr{X}_{\mathscr{Y}}$, or w, diminishes from zero to negative values, at least locally; the physical process is one of compression.

For α near to zero $\mathscr{X}$ can be written as a series:

$$\mathscr{X} = \sum_{n=1} \alpha^{\frac{1}{2}n}\mathscr{X}_n(\mathscr{Y}), \quad \alpha \geqslant 0, \tag{87}$$

where

$$\mathscr{X}_1'' + 2\mathscr{Y}\mathscr{X}_1' = 0, \quad \mathscr{X}_1(0) = 0, \quad \mathscr{X}_1'(0) = -K, \tag{88}$$

$$\mathscr{X}_2'' + 2\mathscr{Y}\mathscr{X}_2' + \tfrac{1}{2}\mathscr{X}_1\mathscr{X}_1'' = 0, \quad \mathscr{X}_2(0) = 1, \quad \mathscr{X}_2'(0) = 0, \tag{89}$$

and similarly for $\mathscr{X}_n$, $n \geqslant 3$. Equations (88) and (89) give

$$\mathscr{X}_1 = -\tfrac{1}{2}\pi^{\frac{1}{2}}K \operatorname{erf}\mathscr{Y}, \tag{90}$$

$$\mathscr{X}_2 = 1 + K^2\frac{\pi^{\frac{1}{2}}}{4}\int_0^{\mathscr{Y}} \left\{(s^2 - \tfrac{1}{2})\operatorname{erf} s + \frac{1}{\pi^{\frac{1}{2}}} s e^{-s^2}\right\} e^{-s^2}\, ds \equiv 1 + \Delta(\mathscr{Y}), \tag{91}$$

so that when α is small an estimate of the shape of a high-frequency characteristic is provided by the approximation

$$\mathscr{X} = \Xi/A = \xi = -\tfrac{1}{2}(\pi\alpha)^{\frac{1}{2}} K \operatorname{erf}\mathscr{Y} + \alpha(1 + \Delta(\mathscr{Y})) + \dots. \tag{92}$$

Equation (92) is valid for $\alpha \geqslant 0$; when $\alpha < 0$, (86) must be used. This is not in the general form (67), for which the shock-fitting formulae (68) and (69) apply, but a modified result which applies especially to (92) and a bow shock wave can be found as follows;

$$\xi_s = \alpha_1 = \alpha_2(1+\Delta(\mathscr{Y}_s)) - f(\alpha_2)\, g(r_s), \tag{93}$$

$$-g(r_s) f^2(\alpha_2) + 2\int_0^{\alpha_1} f(s)\, ds + \alpha_2 f(\alpha_2)\, \Delta(\mathscr{Y}_s) + \int_0^{\alpha_1} \Delta(\mathscr{Y}_s(s)\, [f(s) - sf'(s)]\, ds = 0. \tag{94}$$

Noting that (77) and the special definition of R, namely $A\epsilon^4 r$, make

$$\mathscr{Y} = (\tfrac{1}{2}A\epsilon^4 \beta_{f0} r)^{\frac{1}{2}}, \tag{95}$$

it is readily seen from (93) and (94) that the shock-wave shape is identical with the result in (72), *provided that* δ in that equation is replaced by $A\epsilon^4$ and *provided that* $\Delta(\mathscr{Y})$ *is zero*. Manifestly $\Delta(\mathscr{Y})$ is not zero and (91) shows that it is a rather awkward function of $\mathscr{Y}$ which may, however, be approximated by

$$\Delta(\mathscr{Y}) \simeq \tfrac{1}{12}K^2\mathscr{Y}^4 = \tfrac{1}{48}\beta_{f0}^2 K^2 (A\epsilon^4 r)^2 \tag{96}$$

when $\mathscr{Y}$ is small.

It is clearly not possible to solve (94) for α_2 as a function of $\mathscr{Y}_s$ in any simple manner, on account of the final integral there, but it is possible to estimate the effect of Δ on shock shape by using a mean-value theorem on the integral to give [N.B. $f(\alpha_2) = \alpha_2^{\frac{1}{2}}$, $g(r_s) = \frac{1}{2}\pi^{\frac{1}{2}} K \operatorname{erf} \mathscr{Y}_s$]

$$-g(r_s) + \alpha_2^{\frac{1}{2}}\{\tfrac{4}{3} + \Delta(\mathscr{Y}_s) + \tfrac{1}{3}\Delta(\overline{\mathscr{Y}}_s)\} = 0, \tag{97}$$

where $0 < \overline{\mathscr{Y}}_s < \mathscr{Y}_s$. It follows that the shape of the shock is given approximately by

$$\xi_s \simeq -\tfrac{3}{16} g^2(r_s)\{(1+\Delta(\overline{\mathscr{Y}}_s))\,(1+\tfrac{3}{4}\Delta(\mathscr{Y}_s) + \tfrac{1}{4}\Delta(\overline{\mathscr{Y}}_s))^{-2}\}, \tag{98}$$

which indicates that although it starts at the apex of the cone as a frozen conical shock, as in (72) with $\delta = A\epsilon^4$, its actual shape develops in a rather complicated way with increasing r_s (see figure 2).

We must recall the limitations inherent in (97) and (98), which require α_2 and hence $g^2(r_s)$ to be small; the implication is that $\mathscr{Y}_s \propto \epsilon^2 r_s^{\frac{1}{2}} A^{\frac{1}{2}}$ must be small and, via (96), that the effects of $\Delta(\mathscr{Y})$ on shock shape are themselves small. For example one may suggest as a reasonable guide that the present small-energy shock shape is adequately approximated by (72) with $\delta = A\epsilon^4$ for any $Ar_s \ll \epsilon^{-4}$.

The manner in which the small-energy solution continues for larger values of r cannot be described analytically, so far as we have been able to discover at present. It is possible to adapt Blythe's (1969) analysis to consider the situation in the neighbourhood of some value α_0 (>0) of α and so indeed to show that a discontinuous frozen Rankine–Hugoniot shock must persist to arbitrarily large radii from the nose of the cone. It is however not possible to prove whether this shock eventually evolves into one which decays in the manner described in § 4.2 for intermediate energies, or whether, as physical intuition would suggest, it will behave roughly in this way for 'small' cone angles ϵ, but will persist as a shock of asymptotically constant strength when the cone angle is 'large' enough.

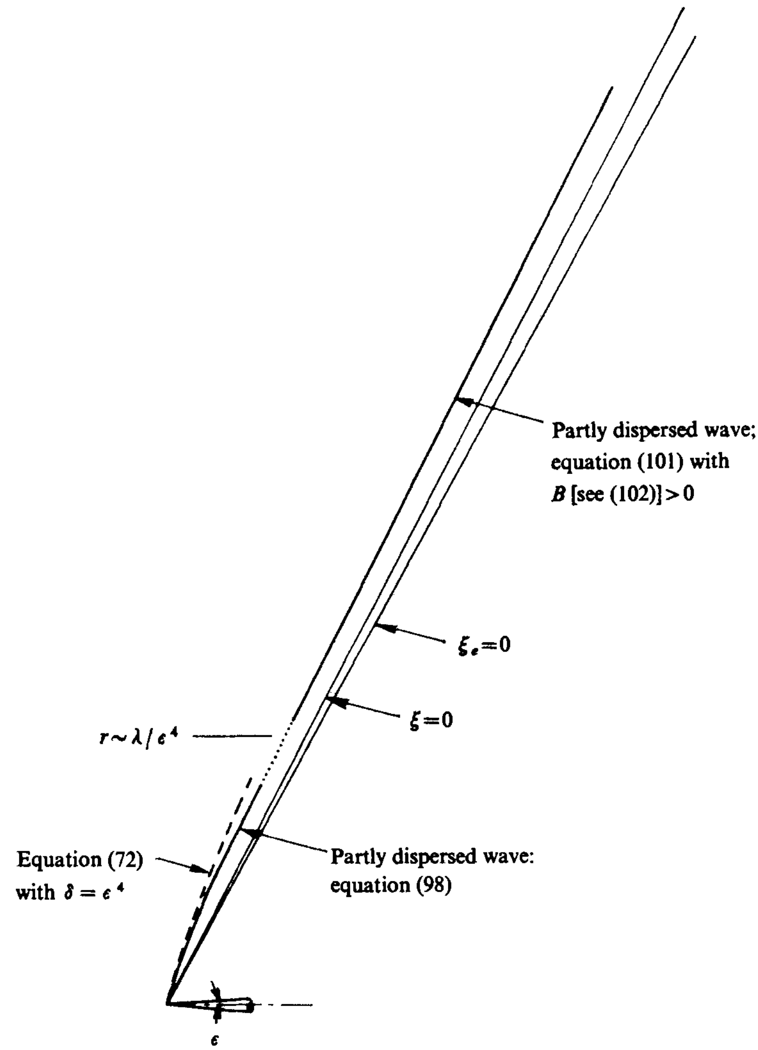

FIGURE 2. The shock wave for small relaxing-mode energy, $\delta = \text{ord}\,\epsilon^4$ (see § 4.3). The part of the wave sketched according to the result in (98) should have its axial distance from $\xi = 0$ reduced to about $\frac{1}{50}$ of the distance depicted, as for figure 1, when $\epsilon \sim 10^{-1}$. The intermediate-energy result (72) is shown for comparison; the small-energy shock wave is more nearly a right circular cone than is the intermediate-energy wave. The shape of the wave in $r < \lambda/\epsilon^4$ is necessarily rather speculative (see § 4.3) as is its axial location for $r > \lambda/\epsilon^4$.

Some evidence that only partly dispersed shocks exist in the present small-energy case can be adduced by seeking what we might describe as a 'fully evolved' solution of (76), which is conveniently expressed in the form

$$U^{(1)}(\Xi, R) = -(\beta_{f0}^2/2M_{f0}^3\,\Gamma_{f0})\,V(\Xi + \beta_{f0}BR), \tag{99}$$

where B is a constant and we recognize that $U^{(1)}$ is most likely to be negative in this compressive situation. Substitution of (99) into (76) shows that $U^{(1)}$ will only depend upon the single 'evolved' conical co-ordinate $\Xi + \beta_{f0} BR$ in an approximate, asymptotic, manner as $\beta_{f0} R \to \infty$. If we ignore the term

$$(V + \beta_{f0} B V')/\beta_{f0} R$$

in this limit we find that V satisfies the equation

$$(VV' - 2BV')' + A^{-1} VV' - (2B/A + 1) V' = 0, \tag{100}$$

approximately. Since V and V' can reasonably be supposed to vanish simultaneously, (100) yields the velocity variation

$$2B \ln V + 2(B+A) \ln \{2(2B+A) - V\} = -(2B/A+1)(\Xi + \beta_{f0} BR) + C, \tag{101}$$

where C is a constant.

Consistent with the view that (101) represents a compression wave, $2(2B+A)$ will be assumed to exceed zero, i.e. $B > -\frac{1}{2}A$; there are then two cases to consider. When $0 > B > -\frac{1}{2}A$, $V \to 0$ as $\Xi + \beta_{f0} BR \to -\infty$ and $V \to 2(2B+A)$ when $\Xi + \beta_{f0} BR \to +\infty$; the wave structure is continuous, indeed fully dispersed, and the wave inclination is, properly, downstream of the undisturbed frozen characteristics. When $B > 0$, $V \to 2(2B+A)$ correctly as $\Xi + \beta_{f0} BR \to +\infty$, but $V \to 0$ for the *same* positive value of the conical co-ordinate; since the solution (101) is therefore two-valued in these circumstances a frozen Rankine–Hugoniot shock must be fitted to the front of the wave system and a partly dispersed wave results; we note that when $B > 0$ the wave described by (101) is inclined upstream of a frozen free-stream characteristic as we should expect for a partly dispersed wave.

We may anticipate that in regions where R is large [see prior to (100)] the motion is substantially in an equilibrium state. Hypothesizing that the waves described in the previous paragraph exist at the head of this equilibrium 'core', it is reasonable to equate the final value of $2(2B+A)$ for V with the relevant equilibrium velocity perturbation immediately downstream of an equilibrium conical shock, namely

$$-3M_{e0}^4 M_{f0} \Gamma_{e0}/\beta_{e0}^2.$$

Recalling the definition of V in (99) we can now find the quantity B from the relation

$$-(\beta_{f0}^2/2M_{f0}^3 \Gamma_{f0})\{2(2B+A)\} = -3M_{e0}^4 M_{f0} \Gamma_{e0}/\beta_{e0}^2.$$

When $\delta = \operatorname{ord} \epsilon^4$ the subscript-e and subscript-f values do not differ markedly and we can therefore write

$$B \simeq \tfrac{1}{2}\{(3M_{f0}^8 \Gamma_{f0}^2/\beta_{f0}^4) - A\}. \tag{102}$$

For any acceptable supersonic stream it is clear that A will have to be numerically very large to make $B \leqslant 0$ and we infer that when $\delta = \operatorname{ord} \epsilon^4$ the bow shock is likely to be always partly dispersed (see figure 2). The reason lies in the fact that in the present case δ is made to diminish with diminishing ϵ; for a fixed δ, reduction of ϵ must eventually lead to an intermediate-energy type of situation, where, as we shall see, fully dispersed waves are essential, the precursor shock having decayed exponentially as described in the previous subsections.

4.4. *Frozen wave head; very small energy*

When $\delta < \operatorname{ord}\epsilon^4$ the relaxing-mode energy is very small and it now becomes necessary to replace (74) and (75) by

$$q = 1+\epsilon^4 Q^{(1)}(\Xi, R) \tag{103}$$

and

$$Q^{(1)} = (\partial q_e/\partial p)_{s,0} P^{(1)}, \tag{104}$$

with $\Xi = \xi$ and $R = \epsilon^4 r$. The source term W now contributes a negligible amount, of relative order ϵ^a $(a > 0)$ if $\delta = O(\epsilon^{4+a})$, to (10), which therefore reduces to the simple statement that

$$2U_R^{(1)} + R^{-1}U^{(1)} + 2M_{f0}^3 \beta_{f0}^{-1} \Gamma_{f0} U^{(1)} U_\Xi^{(1)} = 0. \tag{105}$$

The velocity u is again given by (54), and (105) describes a fully frozen flow; $U^{(1)}$ is easily found and (56) and (104) enable one to calculate the perturbation to the non-equilibrium variable. Relaxation is relegated to the role of a higher-order perturbation to a basic frozen flow. We shall not pursue this case any further.

4.5. *Equilibrium wave head*

The analysis so far has concentrated on the wave motion in regions where $r\delta$, for $\delta \geqslant \operatorname{ord}\epsilon^4$, or $r\epsilon^4$, for $\delta \leqslant \operatorname{ord}\epsilon^4$, is $O(1)$. When $r\delta$ is large the mid-field estimates (27) or (28) are no longer appropriate and we must turn to (47) for relevant information.

Recalling that we are discussing the cone, so that $\mathscr{W}(y) = 2y^{\frac{1}{2}}$ and $\lambda = 1$, (47) can be put in the form

$$u^{(1)} = -\frac{M_{f0}}{\pi^{\frac{1}{2}}}\left(\frac{\delta b^{-2}}{2\beta_{e0} r}\right)^{\frac{1}{4}} 2\int_0^\infty Y^{\frac{1}{2}} \exp\left[-(\zeta_e - Y)^2\right] dY, \tag{106}$$

where

$$\zeta_e = \xi_e/(2\beta_{e0} r\delta b^{-2})^{\frac{1}{2}}. \tag{107}$$

Evidently the low-frequency wavelets are now important. Their location can be found from (49*a*) with the subscript f replaced throughout by the subscript e. Subsequent algebra follows (49*b*)–(52) very closely, with only minor changes to allow for appropriate equilibrium relations. It is found that the relevant equilibrium wavelet is given by

$$\frac{dr}{dx} \simeq \frac{1}{\beta_{e0}}\left\{1 - \epsilon^2 \Gamma_{e0} \frac{M_{e0}^3}{\beta_{e0}^2} u^{(1)}\right\}$$

and since (106) shows that $u^{(1)}$ behaves like $(\delta/r)^{\frac{1}{4}}$ for any fixed ζ_e (such as zero) one readily concludes that the equilibrium wavelet shape is ξ_e – constant $\propto \epsilon^2\delta^{\frac{1}{4}} r^{\frac{3}{4}}$. Since the linear-theory wavelet would be ξ_e = constant it follows that the accumulating error in the mid-field theory is of order $\epsilon^2\delta^{\frac{1}{4}} r^{\frac{3}{4}}$, which is unacceptably bad when it is of the order of ξ_e itself. To reconcile the two estimates, $\xi_e \approx \epsilon^2\delta^{\frac{1}{4}} r^{\frac{3}{4}}$ and $\xi_e^2 \approx r\delta$ (in order to keep ζ_e fixed), it is necessary to have $r \approx \delta/\epsilon^8$ and $\xi_e \approx \delta/\epsilon^4$, so that the new co-ordinates Ξ_e and R_e are defined by

$$\Xi_e = \xi_e \epsilon^4/\delta, \quad R_e = r\epsilon^8/\delta. \tag{108}$$

The dependent variables must be written as

$$u = M_{f0} + \epsilon^4 U_e^{(1)}(\Xi_e, R_e), \tag{109}$$

$$p = (p_0'/\rho_0' a_{f0}'^2) - \epsilon^4 M_{f0} U_e^{(1)}, \tag{110}$$

$$v = -\epsilon^4 \beta_{e0} U_e^{(1)}, \tag{111}$$

$$q_e = 1 + \epsilon^4 (\partial q_e/\partial p)_{s,0}(-M_{f0} U_e^{(1)}) = 1 + \epsilon^4 Q_e^{(1)}. \tag{112}$$

$$q = 1 + \epsilon^4 Q^{(1)} \simeq 1 + \epsilon^4 Q_e^{(1)}, \tag{113}$$

and we note $q = q_e$, to first order, *provided* that $\delta > \operatorname{ord} \epsilon^4$. The difference $q - q_e$ is more important than q or q_e separately in this near-equilibrium situation and it is therefore expedient to define

$$q - q_e = \bar{q} = \epsilon^8 \delta^{-1} \bar{Q}^{(1)}. \tag{114}$$

It follows from (112)–(114) that the relaxation equation (4) becomes

$$\bar{Q}^{(1)} = -Q_{e\Xi_e}^{(1)} = (\partial q_e/\partial p)_{s,0} M_{f0} U_{e\Xi_e}^{(1)}. \tag{115}$$

It is now possible to evaluate the quantities ρa_f^2 and σ in (3) at the local equilibrium state defined by (113) in the knowledge that the relative error in so doing is, from (114), $O(\epsilon^8/\delta)$. It follows, using the local as opposed to the free-stream version of (14), that (3) can now be rewritten in the form

$$(\mathbf{u}\cdot\nabla)p + \rho a_e^2 \nabla\cdot\mathbf{u} + \rho a_e^2 \sigma_e (\mathbf{u}\cdot\nabla)\bar{q} \simeq 0 \tag{116}$$

with the same relative error; a_e is the local (i.e. variable) equilibrium sound speed and it is necessary to use (8) to show that no terms involving entropy variations appear. Substitution of (116) for (3) in the set of equations (1)–(4) makes it expedient to treat these equations as if the characteristics were based on a_e and not a_f. The result is a pair of equations identical with (10) in every way except for the replacement of the subscript f by the subscript e and replacement of W, defined in (11), by

$$W_e = \rho a_e^2 \sigma_e (\mathbf{u}\cdot\nabla)\bar{g}. \tag{117}$$

The necessity of following local changes in the equilibrium sound speed introduces Γ_e, defined in the same way as Γ_f in (52) except that $q = q_e$ during the differentiation instead of being held at a constant value.

Making use of all the foregoing results, and after a little algebra, it is found that $U_e^{(1)}$ satisfies the following axisymmetric form of Burgers' equation:

$$2\frac{\Gamma_{e0} M_{e0}^4}{\beta_{e0} M_{f0}} U_e^{(1)} U_{e\Xi_e}^{(1)} + 2U_{eR_e}^{(1)} + \frac{1}{R_e} U_e^{(1)} = \frac{\beta_{f0}^2}{\beta_{e0}} U_{e\Xi_e\Xi_e}^{(1)}. \tag{118}$$

In reiterating that (118) is valid only when $\delta > \operatorname{ord} \epsilon^4$ we observe that the errors in (47), on which the derivation of (118) depends, are $O(\epsilon^8/\beta_{f0}\delta^2 R_e)$ and

$$O(\Xi_e \epsilon^4/\beta_{f0} R_e \delta)$$

and so are negligible in the new co-ordinate system only when δ obeys the above restriction. Since (47) will provide a boundary condition for $U_e^{(1)}$ in (118) by matching, these observations are important.

It is useful to note that (106) can be evaluated in terms of the parabolic cylinder function $\mathscr{U}(1, -2^{\frac{1}{2}}\zeta_e)$ defined by Abramowitz & Stegun (1965, p. 687). The result is

$$u^{(1)}(\xi, r) = -\tfrac{1}{2}M_{f0}\left(\frac{\delta b^{-2}}{\beta_{f0} r}\right)^{\frac{1}{4}} \exp\left(-\tfrac{1}{2}\zeta_e^2\right) \mathscr{U}(1, -2^{\frac{1}{2}}\zeta_e), \tag{119}$$

and it follows that (118) must be solved subject to the condition that $U_e^{(1)}$ behaves like

$$-\tfrac{1}{2}M_{f0}\left(\frac{b^{-2}}{\beta_{e0} R_e}\right)^{\frac{1}{4}} \exp\left(-\Xi_e^2/4\beta_{e0} R_e b^{-2}\right) \mathscr{U}(1, -\Xi_e/(\beta_{e0} R_e b^{-2})^{\frac{1}{2}}) \tag{120}$$

as $R_e \to 0$ with $\Xi_e/R_e^{\frac{1}{2}}$ fixed.

The axisymmetric Burgers equation does not appear to possess an equivalent to the linearizing Cole–Hopf transformation, which reduces the planar version to a related diffusion equation. This transformation has been used by Parker (1975, private communication) to produce an equation which has the nonlinearity transferred to a different type of term and which permitted him to use a subsequent linearization. The need to match the solution of (118) with the mid-field result unfortunately invalidates this procedure in the present case (Sinai 1975). Chong & Sirovich (1973) have mentioned a similarity transformation of (118); the equation can be used to describe flow past shapes more general than the cone and it has been shown that there is a body of general ogival shape which supports the similarity field; it has a blunted nose and so, strictly, violates the basic slender-body hypothesis, but the blunting is slight and the solution, which is fully discussed elsewhere (Sinai 1976), may well be useful.

The difficult task of finding a solution of (118) subject to condition (120) means that it is worth looking at special situations such, for example, as the analogue of the 'fully evolved' solution of § 4.3. Writing

$$U_e^{(1)}(\Xi_e, R_e) = -V(\Xi_e + B_e R_e), \tag{121}$$

rather as in (99), we find, first, that (121) is valid only if V/R_e is negligible and, second, that

$$(\beta_{f0}^2/2B_e\beta_{e0}) \ln\{V(V - 2B_e M_{f0}\beta_{e0}/\Gamma_{e0} M_{e0}^4)^{-1}\} = \Xi_e + B_e R_e + \text{constant}. \tag{122}$$

Clearly $\Xi_e + B_e R_e \to \mp\infty$ as $V \to 0$ and as $V \to 2B_e M_{f0}\beta_{e0}/\Gamma_{e0} M_{e0}^4$, respectively; joining the compression-wave solution to a downstream equilibrium flow, as in § 4.3, makes B_e equal to $3M_{e0}^8\Gamma_{e0}^2/2\beta_{e0}^3$ and the wave is evidently *essentially* fully dispersed (see figure 1).

Of course we have not *proved* that the compression wave far from the cone evolves into the form (122). Some further evidence in favour of the view that this is the case comes from the fact that when $\Xi_e/R_e^{\frac{1}{2}}$ is large and positive the asymptotic form for $\mathscr{U}$ makes the matching condition (120) read

$$-M_{f0}(2\Xi_e/\beta_{e0} R_e)^{\frac{1}{2}}. \tag{123}$$

This is the proper matching condition for the full equilibrium flow past a cone, which ultimately leads to the value of B_e quoted in the previous paragraph. Since it will be an equilibrium characteristic 'far downstream' from the nose of the cone (i.e. 'far' in the sense that $\Xi_e/R_e^{\frac{1}{2}} \gg 1$ in this case) that intersects the rear of the bow compression wave, the evidence in favour of the fully dispersed wave (122) is improved.

It must be remembered that all lengths in the cone problem are measured in units of the relaxation length, so that 'far' means distant in terms of this particular measure. It is of course possible to revert to the general form of (47) and so to restore λ as a parameter which may take on various extreme magnitudes, always subject to preservation of the errors inherent in (47) as small numbers. This can be done for the particular result (106) for the cone by replacing δ by $\lambda\delta$ in both (106) and (107) and hence also in (119). A formal use of the limit $\lambda \to 0$ with all other quantities fixed then shows that $u^{(1)}$ is exponentially small for $\xi_e < 0$ while

$$u^{(1)} \simeq -M_{f0}\left(\frac{2\xi_e}{\beta_{e0}r}\right)^{\frac{1}{2}}\left\{1-\frac{\beta_{e0}r\lambda\delta b^{-2}}{\xi_e^2}+\ldots\right\} \quad \text{for} \quad \xi_e > 0. \tag{124}$$

The near-equilibrium character of the flow is evident from the first term in (124) and the linear mid-field theory breaks down in respect of its prediction of wavelet position when $\xi_e/r = O(\epsilon^4)$ (the situation is analogous to the one discussed in §4.2 for near-frozen conditions). If (124) is to be a valid representation of $u^{(1)}$ in regions where $\xi_e/r = O(\epsilon^4)$ the restriction $\xi_e/\delta r \ll 1$, which must be met by (47) in general, requires $\delta > \operatorname{ord}\epsilon^4$ and it is only necessary to ensure that $r\delta/\lambda \gg 1$ and, for validity of the asymptotic form (124), $\xi_e/(r\lambda\delta)^{\frac{1}{2}} \gg 1$. If ξ_e is $O(\delta/\epsilon^4)$ and r is therefore $O(\delta/\epsilon^8)$, it is easily shown that the required conditions are met if $\lambda = O(\epsilon^n)$, $n > 0$. This means that, with λ restored as an explicit (small) parameter in the problem, we should again use co-ordinates (108) to describe the nonlinear wave behaviour; (109)–(113) are similarly valid in the new situation, but (114) must be modified to read

$$q - q_e = \bar{q} = \lambda\epsilon^8\delta^{-1}\bar{Q}^{(1)} \tag{125}$$

with the net result that $U_e^{(1)}$ now satisfies (118) modified to the extent that λ multiplies the double Ξ_e derivative on the right-hand side.

If we define
$$W = R_e^{\frac{1}{2}}U_e^{(1)}, \tag{126}$$

$$K_e = \Gamma_{e0}M_{e0}^4/M_{f0}\beta_{e0}, \quad C_e = \beta_{f0}^4/2\beta_{e0}, \tag{127}$$

then W satisfies the equation

$$K_e R_e^{-\frac{1}{2}} W W_{\Xi_e} + W_{R_e} = \lambda C_e W_{\Xi_e\Xi_e}, \tag{128}$$

subject to the condition

$$W \to -M_{f0}\left(\frac{2\Xi_e}{\beta_{e0}}\right)^{\frac{1}{2}}\left\{1-\frac{\beta_{e0}R_e b^{-2}}{\Xi_e^2}\lambda+\ldots\right\} \tag{129}$$

as $R_e \to 0$ with $\Xi_e/R_e^{\frac{1}{2}}$ fixed.

If A is a new variable, defined such that

$$(\partial\Xi_e/\partial R_e)_A = K_e R_e^{-\frac{1}{2}} W, \tag{130}$$

(128) can be rewritten in the form

$$\left(\frac{\partial W}{\partial R_e}\right)_A = \lambda C_e\left\{\frac{\partial^2 W}{\partial A^2}\left(\frac{\partial A}{\partial \Xi_e}\right)^2+\frac{\partial W}{\partial A}\left(\frac{\partial^2 A}{\partial \Xi_e^2}\right)\right\}. \tag{131}$$

Since $\lambda = O(\epsilon^n)$, $n > 0$, is essentially small a solution of (131) is sought in the form

$$W = W^{(1)} + \lambda W^{(2)} + \dots. \tag{132}$$

The solution for $W^{(1)}$ is

$$W^{(1)} = -M_{f0}\left(\frac{2A}{\beta_{e0}}\right)^{\frac{1}{2}}, \quad \Xi_e = A - 2M_{f0} K_e \left(\frac{2AR_e}{\beta_{e0}}\right)^{\frac{1}{2}}, \tag{133}$$

where the arbitrary function of A appearing in the integral of (130) has been set equal to A. We observe that $A \to \Xi_e$ under the matching condition quoted after (129). It is not difficult to continue with the evaluation of W by calculating $W^{(2)}$ and thus, via (130), to find a revised relation between Ξ_e, R_e and A, but we shall not continue with this exercise here beyond making the point that the second term in W can be shown to behave like the second term in (129), as it must.

Solution (133) is multi-valued in the physical, Ξ_e, R_e plane, and requires the introduction of a shock wave at the head of the flow field. Since (133) is like (67), the shock-fitting rules (68) and (69) can be used to show that this shock must be conical, with vertex at the vertex of the body in the present limit of vanishing λ. The implication is that $\lambda C_e W_{\Xi_e \Xi_e}$ is indeterminate at the shock wave, the problem is evidently of the singular perturbation variety, and the approximation must be corrected to account for this. The conical shock can be shown to lie along $\Xi' = 0$, where

$$\Xi' = \Xi_e + 3M_{e0}^8 \Gamma_{e0}^2 R_e / 2\beta_{e0}^2, \tag{134}$$

and we note in passing the correspondence with the 'evolved' solution (121) and (122). Changing to co-ordinates Ξ' and R_e and stretching Ξ' by the factor $1/\lambda$ converts (128) into an equation whose solution is precisely (122) with the right-hand side divided by λ, provided that terms in λ/R_e are negligible. The validity of the $W^{(1)}$ solution depends on $\lambda W^{(2)}$ being satisfactorily negligible and some idea of the criterion for this can be seen in (129); with Ξ_e near to the shock wave, $\lambda/R_e \ll 1$ proves once again to be the correct condition and it appears that the fully dispersed wave should be more-or-less fully established when R_e is, say, 10λ. Recalling the definition of R_e from (108), this requires r/λ to be $10\delta\epsilon^{-8}$. Thus, writing $\delta = O(\epsilon^{4-m})$, $m > 0$, it is clear that the development distance for the fully dispersed wave will be very large in units of the relaxation length, even when the relaxing-mode energy is small, i.e. $m \to 0$ (see figure 1).

The introduction of λ as an explicit parameter has made it possible to check the validity of (121) and (122) but, more important, it also shows how to improve on these estimates via calculation of $W^{(2)}$, etc. So far we have only the single restriction $\lambda = O(\epsilon^n)$, $n > 0$; the asymptotic mid-field solution (124) shows that if $\lambda < \operatorname{ord} \epsilon^2$ the second term becomes so small that it must be included with the second-order $[O(\epsilon^4)]$ mid-field terms. The near-equilibrium relaxation processes then appear as higher-order modifications to a basically equilibrium background flow. The far-field nonlinear wave region will still include first-order relaxation effects unless $\lambda \leqslant \operatorname{ord} \epsilon^4$, when they will again be demoted to second-order importance. The characteristic length L' for a cone is of course an artificial parameter, so that it is always possible to arrange for λ to lie in the interval between 1 and ϵ^2. This means that we are arbitrarily selecting regions of the flow where relaxation effects are important. When the body has a definite value of L' we shall no longer be permitted the luxury of choosing λ.

5. Conclusion

A list of the main conclusions from the present work must include the following.

The general linear mid-field behaviour in regions where its predictions begin to break down has been established, together with the important error bounds.

For the flow past a cone the asymptotic ($r \to \infty$) shock-wave structure is known when ϵ, δ and M_{f0} are given, but in order to trace its evolution it is necessary to distinguish between three separate relaxing-mode energy levels. When $\delta <$ ord ϵ^4 relaxation becomes an essentially higher-order effect. When ord $1 \geqslant \delta >$ ord ϵ^4 the wave head is a frozen Rankine–Hugoniot shock, whose shape is given by (72), provided that $r/\lambda \lesssim 1/\delta$; when $r/\lambda \gg 1/\delta$ the main compression front is essentially fully dispersed, but the evidence is that the wave is not fully established until $r/\lambda \gg \delta\epsilon^{-8}$; this is a large number of relaxation lengths, even when the energy level δ is small. When $\delta =$ ord ϵ^4 the wave head is again a frozen Rankine–Hugoniot shock whose shape is given reasonably well by (72) when $r/\lambda \ll$ ord ϵ^{-4}. For larger r/λ the evidence favours persistence of this situation in the form of a partly dispersed wave; a fully dispersed wave is unlikely.

The weakness of far-field disturbances makes it quite probable that δ will exceed ord ϵ^4 in practical situations. Figures 1 and 2 illustrate the features mentioned above.

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